Euler-Poincaré equations for anelastic fluid flows

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Abstract

We show that the ideal (nondissipative) form of the dynamical equations for the Lipps-Hemler formulation of the anelastic fluid model follow as Euler-Poincaré equations, obtained from a constrained Hamilton's principle expressed in the Eulerian fluid description. This establishes the mathematical framework for the following properties of these anelastic equations: the Kelvin-Noether circulation theorem, conservation of potential vorticity on fluid parcels, and the Lie-Poisson Hamiltonian formulation possessing conserved Casimirs, conserved domain integrated energy and an associated variational principle satisfied by the equilibrium solutions. We then introduce a modified set of anelastic equations that represent the mean anelastic motion, averaged over subgrid scale rapid fluctuations, while preserving the mathematical properties of the Euler-Poincaré framework.

1 Introduction

The Eulerian formulation of the action principle for an ideal fluid casts it into a form that is amenable to asymptotic expansions and thereby facilitates the creation and analysis of approximate fluid theories. Such an Eulerian action principle results whenever the general theory of reduction by symmetry groups is applied to Lagrangian systems, thereby yielding ***Euler–Poincaré equations***, the Lagrangian analog of Lie-Poisson Hamiltonian equations, Marsden and Ratiu [1994]. This Euler–Poincaré setting provides a shared mathematical structure for many problems in geophysical fluid dynamics (GFD), with several benefits, both immediate (such as a systematic approach to hierarchical modeling and versions of Kelvin's circulation theorem for these models) and longer term (e.g., structured multisymplectic integration algorithms).

For example, by using the Euler–Poincaré approach, Holm, Marsden and Ratiu [1998a,b], find that the action principles of a variety of incompressible fluid models for GFD are related by different levels of truncation of asymptotic expansions and velocity-pressure decompositions, as applied in Hamilton's principle for the unapproximated Euler equations of rotating stratified ideal incompressible fluid dynamics. This sequence of GFD models includes the Euler equations themselves, followed by their approximations, namely: Euler-Boussinesq equations (EB), primitive equations (PE), Hamiltonian balance equations (HBE), and generalized Lagrangian mean (GLM) equations. It also includes rotating shallow water equations (RSW), semigeostrophic equations (SG), and quasigeostrophic equations (QG). Thus, asymptotic expansions and velocity-pressure

decompositions of Hamilton's principle for the Euler equations describing the motion of a rotating stratified ideal incompressible fluid are used in Holm, Marsden and Ratiu [1998a,b] to cast the standard models of GFD into Euler-Poincaré form and thereby unify these descriptions and their properties at various levels of approximation. For related developments and additional structure preserving approximations constructed from a similar viewpoint, see Allen and Holm [1996], Allen, Holm and Newberger [1998] and Holm and Zeitlin [1998].

Recently, Bannon [1996] reexamined the anelastic approximation for deep fluid convection and proposed an alternative form of the anelastic equations. This alternative model combines the results of Dutton and Fichtl [1969] and Lipps and Hemler [1982] to produce a hybrid theory that (1) conserves the domain integrated energy; (2) preserves potential vorticity on fluid parcels; and (3) accurately represents the acoustic adjustment process in Lamb's problem. The equations for a dry anelastic compressible fluid (atmosphere) rotating at angular frequency $\mathbf{\Omega}$ under constant vertical gravitational acceleration $g\hat{\mathbf{z}}$ take the following form Bannon [1996]

$$\begin{aligned}
\frac{d\mathbf{u}}{dt} + 2\mathbf{\Omega}\times\mathbf{u} &= -\nabla\left(\frac{p'}{\rho_s}\right) + \frac{g\theta'}{\theta_s}\hat{\mathbf{z}}\,, && (1.1)\\
\nabla\cdot(\rho_s\mathbf{u}) &= 0\,, && (1.2)\\
\frac{d(\theta_s+\theta')}{dt} &= 0\,, && (1.3)\\
\frac{\theta'}{\theta_s} &= \frac{p'}{\rho_s g H_\rho} - \frac{\rho'}{\rho_s}\,, && (1.4)\\
\frac{p'}{p_s} &= \frac{\rho'}{\rho_s} + \frac{T'}{T_s}\,. && (1.5)
\end{aligned}$$

In these equations, the fluid velocity is denoted $\mathbf{u}$, the advective time derivative is $d/dt = \partial/\partial t + \mathbf{u}\cdot\nabla$, and the state variables for this anelastic motion are: pressure p, density ρ, specific entropy θ, and temperature T. These state variables consist of the sum of the base state (with subscript s) and a dynamic contribution denoted with a prime, as in

$$\theta(x,y,z,t) = \theta_s(z) + \theta'(x,y,z,t)\,, \tag{1.6}$$

where θ' is the dynamic contribution to the specific entropy field. The base state is taken to satisfy

$$\frac{dp_s}{dz} = -g\rho_s\,, \quad p_s = \rho_s R T_s\,, \quad C_p\,\theta_s\frac{d\pi_s}{dz} = -\,g\,, \tag{1.7}$$

where $\pi_s = T_s/\theta_s$. The constants R and C_p are the ideal gas constant R and the specific heat at constant pressure C_p for dry air. The density scale height is given by $1/H_\rho = -\rho_s(z)^{-1}d\rho_s/dz$. Given the base state functions satisfying relations (1.7), as well as the velocity $\mathbf{u}$ and the dynamic contributions p' and θ' at any time, the thermodynamic diagnostic relations (1.4) – (1.5) complete the description. The distinctions between this anelastic model and the traditional models Dutton and Fichtl [1969], Lipps and Hemler [1982] are discussed in detail by Bannon [1996]. For our purposes here, the important point is that the dynamical equations (1.1) - (1.3) agree in the formulations of both Lipps and Hemler [1982] and Bannon [1996]

In this paper, we show that the dynamical equations (1.1) – (1.3) for the ideal (nondissipative) anelastic model follow as Euler-Poincaré equations, obtained from a constrained Hamilton's principle expressed in the Eulerian description. We then introduce a modified set of anelastic equations that represent the mean anelastic motion, averaged over subgrid scale rapid fluctuations, while

preserving the mathematical properties of the Euler-Poincaré framework. Euler-Poincaré equations are the Lagrangian analog of Lie-Poisson Hamiltonian systems Marsden and Ratiu [1994], Holm, Marsden and Ratiu [1998a,b]. Among other things, the Euler-Poincaré formulation of the anelastic fluid equations provides their Kelvin-Noether circulation theorem. This theorem is the basis for the conservation of anelastic potential vorticity on fluid parcels. Domain-integrated energy is also conserved and the relation of the Euler–Poincaré equations to the Lie-Poisson Hamiltonian formulation of the anelastic dynamics is given by a Legendre transformation at the level of the vector fields satisfying the weighted divergenceless condition (1.2). The Casimir conservation laws for this Lie-Poisson Hamiltonian formulation provide a constrained-energy variational principle for the equilibrium solutions of anelastic dynamics and form a basis for determining their Lyapunov stability conditions, as done for the Euler-Boussinesq equations in Abarbanel et al. [1986]. (The Euler-Boussinesq equations form a special case of the anelastic model in which the base state is constant.)

In related previous work, a two-dimensional study of the Hamiltonian structure of the Lipps-Hemler anelastic model was presented by Scinocca and Shepherd [1992], who also studied wave-activity conservation laws in the two-dimensional case. A canonical Hamiltonian formulation of the Lipps-Hemler anelastic model in three-dimensions was given in Bernardet [1995], Appendix A, for the Lagrangian fluid description of these equations. Perhaps not unexpectedly, the Lie-Poisson Hamiltonian formulation for the Eulerian fluid description of these equations agrees the canonical formulation of Bernardet [1995] and provides an alternative perspective. Indeed it must, as the Euler-Poincaré theorem Holm, Marsden and Ratiu [1998a] proves for a class of Hamilton's principles that includes ideal continuum dynamics that the following four dynamical perspectives of fluid mechanics are equivalent: Hamilton's principle for the Lagrangian fluid description; the Euler-Lagrange equations in the Lagrangian fluid description; Hamilton's principle for the Eulerian fluid description with certain constrained variations, similar to those for reduced Lagrange d'Alembert equations; and the Euler–Poincaré equations in the Eulerian fluid description.

The methods of this paper are based on reduction of variational principles; that is, on Lagrangian reduction. See Cendra et al. [1987, 1998a,b] and Marsden and Scheurle [1993a,b]), who also discuss systems with nonholonomic constraints. The latter has been demonstrated in the work of Bloch, Krishnaprasad, Marsden and Murray Bloch et al. [1996], who derived the reduced Lagrange d'Alembert equations for such nonholonomic systems. Coupled with the methods of the present paper, these techniques for handling nonholonomic constraints would also be useful, if required, in continuum systems. In addition, it seems likely that applications of the techniques of multisymplectic geometry to numerical integrators associated with multisymplectic reduction will be exciting developments for the present setting; see Marsden et al. [1998] for a discussion of this approach.

Organization of the Paper. In §2 we recall from Holm, Marsden and Ratiu [1998a] the results of the Euler-Poincaré theorem for Lagrangians in continuum mechanics depending on advected parameters along with their associated Kelvin–Noether theorem and Lie-Poisson Hamiltonian formulation. These results establish the mathematical framework into which we place the dynamical equations for the anelastic model in §3. In §4 we introduce a modified set of anelastic equations that represent the mean anelastic motion, averaged over subgrid scale rapid fluctuations of amplitude α, while preserving the mathematical properties of the Euler-Poincaré framework.

2 Applications of the Euler–Poincaré Theorem in GFD

Here we recall from Holm, Marsden and Ratiu [1998a] the statements of the Euler–Poincaré equations and their associated Kelvin–Noether theorem in the context of continuum mechanics and approximate models in geophysical fluid dynamics.

The Euler-Poincaré equations for a GFD Lagrangian $L\,[\mathbf{u}, D, b\,]$ involve fluid velocity $\mathbf{u}$, buoyancy (or specific entropy) b and density D as functions of three dimensional space with coordinates $\mathbf{x}$ and time t. In vector notation, these equations are expressed as Holm, Marsden and Ratiu [1986, 1998a,b,c] and Holm [1996],

$$\frac{d}{dt}\frac{1}{D}\frac{\delta L}{\delta \mathbf{u}} \;+\; \frac{1}{D}\frac{\delta L}{\delta u^j}\nabla u^j \;+\; \frac{1}{D}\frac{\delta L}{\delta b}\nabla b - \nabla\frac{\delta L}{\delta D} = 0, \tag{2.1}$$

or, equivalently, in "curl form" as,

$$\frac{\partial}{\partial t}\Big(\frac{1}{D}\frac{\delta L}{\delta \mathbf{u}}\Big) \;-\; \mathbf{u}\times \mathrm{curl}\Big(\frac{1}{D}\frac{\delta L}{\delta \mathbf{u}}\Big) \;+\; \nabla\Big(\mathbf{u}\cdot\frac{1}{D}\frac{\delta L}{\delta \mathbf{u}} \;-\; \frac{\delta L}{\delta D}\Big) \;+\; \frac{1}{D}\frac{\delta L}{\delta b}\nabla b = 0\,. \tag{2.2}$$

The Euler–Poincaré system is completed by including the auxiliary equations for advection of the buoyancy (or specific entropy) b,

$$\frac{\partial b}{\partial t} + \mathbf{u}\cdot\nabla b = 0\,, \tag{2.3}$$

and the continuity equation for the density D,

$$\frac{\partial D}{\partial t} + \nabla\cdot(D\mathbf{u}) = 0\,. \tag{2.4}$$

For incompressible flows, one sets $D = 1$ in the continuity equation, so that $\nabla\cdot\mathbf{u} = 0$. For anelastic flows, one sets $D = \rho_s(z)$ in the continuity equation with a prescribed stably stratified reference density profile $\rho_s(z)$, so that $\nabla\cdot(\rho_s(z)\mathbf{u}) = 0$.

The Euler–Poincaré motion equation in either form (2.1) or (2.2) results in the ***Kelvin-Noether circulation theorem***,

$$\frac{d}{dt}\oint_{\gamma_t(\mathbf{u})}\frac{1}{D}\frac{\delta L}{\delta \mathbf{u}}\cdot d\mathbf{x} = -\oint_{\gamma_t(\mathbf{u})}\frac{1}{D}\frac{\delta L}{\delta b}\nabla b\cdot d\mathbf{x}\,, \tag{2.5}$$

where the curve $\gamma_t(\mathbf{u})$ moves with the fluid velocity $\mathbf{u}$. Then, by Stokes' theorem, the Euler–Poincaré equations generate circulation of the quantity $D^{-1}\delta L/\delta\mathbf{u}$ whenever the gradients ∇b and $\nabla(D^{-1}\delta L/\delta b)$ are not collinear.

Taking the curl of equation (2.2) and using advection of the buoyancy b and the continuity equation for the density D yields ***conservation of potential vorticity on fluid parcels***, as expressed by

$$\frac{\partial q}{\partial t} + \mathbf{u}\cdot\nabla q = 0\,, \quad \text{where} \quad q \equiv \frac{1}{D}\nabla b\cdot\mathrm{curl}\left(\frac{1}{D}\frac{\delta L}{\delta \mathbf{u}}\right). \tag{2.6}$$

Consequently, the following domain integrated quantities are conserved, for any function Φ,

$$C_\Phi = \int d^3x\; D\,\Phi(b,q)\,, \quad \forall\,\Phi\,. \tag{2.7}$$

The absence of explicit time dependence in the Lagrangian $L\,[\mathbf{u}, D, b\,]$ gives the ***conserved domain integrated energy***, via Noether's theorem for time translation invariance. This energy is easily calculated using the ***Legendre transform*** to be

$$E\,[\mathbf{u}, D, b\,] = \int d^3x\, \left(\mathbf{u}\cdot\frac{\delta L}{\delta \mathbf{u}}\right) - L\,[\mathbf{u}, D, b\,]\,. \tag{2.8}$$

When the Legendre transform is completed to express $E\,[\mathbf{u}, D, b\,]$ as $H\,[\mathbf{m}, D, b\,]$ with $\mathbf{m} \equiv \delta L/\delta \mathbf{u}$ and $\delta H/\delta \mathbf{m} = \mathbf{u}$, the Euler–Poincaré system (2.1)–(2.4) may be expressed in Hamiltonian form

$$\frac{\partial \mu}{\partial t} = \{\mu, H\}\,, \quad \text{with} \quad \mu \in [\mathbf{m}, D, b\,]\,, \tag{2.9}$$

and ***Lie-Poisson bracket*** given in Euclidean component form by

$$\begin{aligned}\{F, H\}[\mathbf{m}, D, b\,] \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (2.10)\\ = \; -\int d^3x\, \Big\{ \frac{\delta F}{\delta m_i}\Big[\big(\partial_j m_i + m_j\partial_i\,\big)\frac{\delta H}{\delta m_j} \;+\; \big(D\partial_i\,\big)\frac{\delta H}{\delta D} \;-\; \big(b_{,i}\,\big)\frac{\delta H}{\delta b}\Big]\\ + \frac{\delta F}{\delta D}\big(\partial_j D\big)\frac{\delta H}{\delta m_j} \;+\; \frac{\delta F}{\delta b}\big(b_{,j}\big)\frac{\delta H}{\delta m_j}\Big\}\,.\end{aligned}$$

The conserved quantities C_Φ in (2.7) are then understood in the ***Lie-Poisson Hamiltonian formulation*** (2.9) – (2.10) of the Euler–Poincaré system (2.1) – (2.4) as ***Casimirs*** that commute under the Lie-Poisson bracket (2.10) with any functional of $[\mathbf{m}, D, b\,]$. The Casimirs also result via Noether's theorem from symmetry of the Hamilton's principle for the Euler–Poincaré system under the "particle relabeling transformations" that leave invariant the Lagrangian $L[\mathbf{u}, D, b\,]$. From the viewpoint of Noether's theorem, this particle relabeling symmetry corresponds to invariance of the Hamilton's principle for the Euler–Poincaré equations under the transformation from the Lagrangian to the Eulerian fluid description, by pullback of the right action of the diffeomorphism group on the configuration space of the Lagrangian fluid parcel positions and their velocities. For full mathematical details, consult Marsden and Ratiu [1994]; Holm, Marsden and Ratiu [1998a,b,c].

The four properties (2.5)–(2.8) and the Lie-Poisson Hamiltonian formulation (2.9) – (2.10) of the Euler–Poincaré equation (2.1) and its auxiliary equations (2.3) and (2.4) are desirable elements of approximate models for applications in geophysical fluid dynamics expressed in the variables $[\mathbf{u}, D, b\,]$. Thus, the Euler–Poincaré theory offers a unified framework in which to derive approximate GFD models that possess these properties: the Kelvin-Noether circulation theorem, conservation of potential vorticity on fluid parcels, and the Lie-Poisson Hamiltonian formulation with its associated conserved Casimirs and conserved domain integrated energy. Previous work of Holm, Marsden and Ratiu [1998a,b,c] has shown that many useful GFD approximations may be formulated as Euler–Poincaré equations, whose shared properties thus follow from this underlying common framework. The aim of the next section of this paper is to cast the dynamical anelastic equations (1.1) – (1.3) into the Euler–Poincaré framework, as well.

3 The dynamical anelastic equations are Euler–Poincaré

The Lagrangian

In the Eulerian fluid representation, we consider Hamilton's principle for fluid motion in a three dimensional domain with action functional $\mathcal{S} = \int dt\, L$ and Lagrangian $L[\mathbf{u}, D, b\,]$ given by

$$L = \int d^3x\; D\left(\frac{1}{2}|\mathbf{u}|^2 + \mathbf{u}\cdot\mathbf{R}(\mathbf{x}) - gz - C_p\pi_s(z)b\right) + p'\left(1 - \frac{D}{\rho_s(z)}\right), \tag{3.1}$$

where D is the mass density and $\mathbf{R}(\mathbf{x})$, $\pi_s(z)$, and $\rho_s(z)$ are given functions of their arguments. This Lagrangian produces the following variations at fixed $\mathbf{x}$ and t

$$\begin{aligned}
&\frac{1}{D}\frac{\delta L}{\delta \mathbf{u}} = \mathbf{u} + \mathbf{R}(\mathbf{x})\,, \quad \frac{\delta L}{\delta b} = -C_p\pi_s(z)\,, \quad \frac{\delta L}{\delta p'} = 1 - \frac{D}{\rho_s(z)}\,,\\
&\frac{\delta L}{\delta D} = \frac{1}{2}|\mathbf{u}|^2 + \mathbf{u}\cdot\mathbf{R}(\mathbf{x}) - gz - C_p\pi_s(z)b - \frac{p'}{\rho_s(z)}\,.
\end{aligned} \tag{3.2}$$

Hence, from the Euclidean component formula (2.1) for Hamilton principles of this type, we find the motion equation for such a fluid in three dimensions,

$$\frac{d\mathbf{u}}{dt} - \mathbf{u}\times \mathrm{curl}\mathbf{R} + \nabla\left(\frac{p'}{\rho_s}\right) + \left(g + C_p b\frac{d\pi_s}{dz}\right)\hat{\mathbf{z}} = 0\,, \tag{3.3}$$

where $\mathrm{curl}\,\mathbf{R} = 2\mathbf{\Omega}(\mathbf{x})$ is the Coriolis parameter (i.e., twice the local angular rotation frequency). We use equation (1.7) to rewrite the last term in parentheses as

$$g + C_p b\frac{d\pi_s}{dz} = g\left(1 - \frac{b}{\theta_s}\right) = -g\frac{\theta'(\mathbf{x},t)}{\theta_s(z)}\,, \tag{3.4}$$

in which we identify b as the total specific entropy,

$$b = \theta(\mathbf{x},t) = \theta_s(z) + \theta'(\mathbf{x},t)\,, \tag{3.5}$$

since each satisfies the scalar advection relation (1.3), cf. (2.3). Hence, from (3.3) and (3.4) we recover the anelastic motion equation (1.1), namely,

$$\frac{d\mathbf{u}}{dt} - \mathbf{u}\times 2\mathbf{\Omega}(\mathbf{x}) + \nabla\left(\frac{p'}{\rho_s}\right) - \frac{g\theta'}{\theta_s}\hat{\mathbf{z}} = 0\,, \tag{3.6}$$

as the Euler–Poincaré equation for the Lagrangian (3.1). Finally, we substitute the contraint $D = \rho_s(z)$ obtained from stationarity of the Lagrangian (3.1) with respect to variations in p' into the continuity equation (2.4) to find the anelastic divergence condition $\nabla\cdot\rho_s\mathbf{u} = 0$, i.e., equation (1.2). Preservation of this condition determines the dynamic pressure contribution, p', by solving the elliptic equation obtained by taking the divergence of the anelastic motion equation (3.8) after multiplying it by the base density $\rho_s(z)$,

$$-\Delta p' = g\frac{\partial \rho'}{\partial z} + \mathrm{div}\,(\mathrm{nonlinearity})\,. \tag{3.7}$$

The boundary condition for this elliptic equation is obtained from the normal component of the anelastic motion equation (3.8) evaluated on the boundary and using the boundary condition for

the velocity, e.g., that it has no normal component at the boundary, which yields a Neumann boundary condition for obtaining the pressure.

$$-\frac{\partial p'}{\partial n} = g\rho'(\hat{\mathbf{n}}\cdot\hat{\mathbf{z}}) + \hat{\mathbf{n}}\cdot(\text{nonlinearity})\,. \tag{3.8}$$

See the treatment in Bernardet [1995] for a discussion of alternative velocity boundary conditions for the Lipps-Hemler anelastic model.

The Kelvin–Noether theorem

From equation (2.5), the Kelvin–Noether circulation theorem corresponding to the anelastic motion equation (3.8) for an ideal anelastic fluid in three dimensions is,

$$\frac{d}{dt}\oint_{\gamma_t(\mathbf{u})}(\mathbf{u}+\mathbf{R})\cdot d\mathbf{x} = -\oint_{\gamma_t(\mathbf{u})} C_p\,\theta\,\nabla\pi_s(z)\cdot d\mathbf{x}\,, \tag{3.9}$$

where the curve $\gamma_t(\mathbf{u})$ moves with the anelastic fluid velocity $\mathbf{u}$. By Stokes' theorem, the anelastic equations generate circulation of $(\mathbf{u}+\mathbf{R})$ around $\gamma_t(\mathbf{u})$ whenever the gradient of specific entropy θ has a horizontal component. Using advection of θ and the anelastic divergence condition, one finds conservation of potential vorticity $q_{\rm Anel}$ on fluid parcels, cf. equation (2.6),

$$\frac{\partial q_{\rm Anel}}{\partial t} + \mathbf{u}\cdot\nabla q_{\rm Anel} = 0\,, \quad \text{where} \quad q_{\rm Anel} = \frac{1}{\rho_s(z)}\nabla\theta\cdot\text{curl}\,(\mathbf{u}+\mathbf{R})\,. \tag{3.10}$$

Consequently, the following domain integrated quantities are conserved, for any function Φ, cf. equation (2.7),

$$C_\Phi = \int d^3x\ \rho_s(z)\,\Phi(\theta, q_{\rm Anel})\,, \quad \forall\,\Phi\,. \tag{3.11}$$

Energy conservation, Lie-Poisson Hamiltonian formulation and nonlinear Lyapunov stability analysis

The conserved anelastic energy is easily calculated using the Legendre transform of the Lagrangian (3.1) to be

$$E_{Anel} = \int d^3x\ \rho_s(z)\left(\frac{1}{2}|\mathbf{u}|^2 + gz + C_p\pi_s(z)\theta\right). \tag{3.12}$$

The corresponding Hamiltonian is (with $b=\theta$)

$$H_{Anel} = \int d^3x\ \left(\frac{1}{2D}|\mathbf{m}-D\mathbf{R}|^2 + Dgz + C_p\pi_s(z)\,D\,b\right) + p'\left(\frac{D}{\rho_s(z)}-1\right). \tag{3.13}$$

The Lie-Poisson bracket (2.10) now generates the dynamical anelastic equations (1.1) – (1.3) from this Hamiltonian according to equations (2.9).

The canonical Hamiltonian formulation of the Lipps-Hemler dynamics due to Bernardet [1995], Appendix A, is based on the Hamiltonian in the Lagrangian fluid description,

$$H_{canon} = \int d^3a\ \left(\frac{1}{2}|\dot{\boldsymbol{x}}(\mathbf{a},t)|^2 + C_p\pi_s(z(\mathbf{a},t))\,b(\mathbf{a})\right). \tag{3.14}$$

Transforming this Hamiltonian to the Eulerian fluid description yields,

$$H_{Anel} = \int d^3x \, \left(\frac{1}{2D} |\mathbf{m} - D\mathbf{R}|^2 + C_p \pi_s(z) \, D \, b \right) + p' \left(\frac{D}{\rho_s(z)} - 1 \right) , \tag{3.15}$$

in which we again impose the anelastic density constraint explicitly, by using the pressure p' as a Lagrange multiplier. Applying the Lie-Poisson bracket (2.10) with this Hamiltonian yields the same dynamical anelastic equations as in (1.1) – (1.3), up to a redefinition of pressure to incorporate the gravitational acceleration. Thus, as guaranteed by the Euler-Poincaré theorem Holm, Marsden and Ratiu [1998a] and the general theory of reduction Marsden and Ratiu [1994], the Lie-Poisson Hamiltonian formulation of the dynamical anelastic equations presented here in the Eulerian fluid description is equivalent to the canonical Hamiltonian formulation due to Bernardet [1995] in the Lagrangian fluid description.

In the Eulerian fluid description we use the Casimir conserved quantities (2.7) to find the following ***variational principle for anelastic equilibrium solutions***: The equilibrium solutions of the dynamical anelastic equations occur at critical points of the sum H_Φ, where

$$H_\Phi = H_{Anel} + C_\Phi \, , \tag{3.16}$$

and

$$C_\Phi = \int d^3x \; D \, \Phi(b, q) \, , \quad \forall \, \Phi \, , \quad \text{where} \quad q \equiv \frac{1}{D} \nabla b \cdot \text{curl}\big(\mathbf{m}/D\big). \tag{3.17}$$

Thus, the Casimir conservation laws for this Lie-Poisson Hamiltonian formulation of the three-dimensional anelastic equations in the Eulerian fluid description provide a constrained-energy variational principle for the equilibrium solutions of the dynamical anelastic equations and form a basis for determining their Lyapunov stability conditions, as done for the Euler-Boussinesq equations in Abarbanel et al. [1986]. The Euler-Boussinesq equations form a special case of the dynamical anelastic equations in which the base state is constant. Consequently, the analysis of the nonlinear Lyapunov stability conditions for the equilibrium solutions of the three-dimensional anelastic equations follows a similar procedure to that performed in Abarbanel et al. [1986] and produces a similar result, modulo the nonconstant base state.

The pseudo-incompressible approximation (PIA)

The PIA of Durran [1989] enhances the anelastic equations by allowing the influence of the base specific entropy field $\theta_s(z)$ on the mass balance. Additional studies of the PIA and comparisons of its performance with the dynamics of the anelastic and Euler-Boussinesq models appear in Nance and Durran [1994], Lilly [1994].

As an Euler-Poincaré system, the PIA equations arise from a modification of the anelastic Lagrangian in equation (4.1). Namely, we consider the Lagrangian for the PIA system given by

$$\begin{aligned} \ell_{PIA} \;\; = \;\; & \int d^3x \, \Big[D \Big(\frac{1}{2} |\mathbf{u}|^2 + \mathbf{u} \cdot \mathbf{R}(\mathbf{x}) - gz - C_p \pi_s(z) \, \theta \Big) \\ & + \tilde{p} \underbrace{\Big(\rho_s(z) \theta_s(z) - D \, \theta \Big)}_{\text{PIA constraint}} \Big] . \end{aligned} \tag{3.18}$$

This Lagrangian slightly modifies the volume constraint in the anelastic Lagrangian (4.1). The modified PIA constraint on D imposed by the pressure $\tilde{p}$ requires

$$D^* \equiv D\,\theta = \rho_s(z)\,\theta_s(z)\,.$$

Hence, the fluid velocity $\mathbf{u}$ must satisfy a weighted incompressibility relation (pseudo-incompressibility)

$$\frac{\partial D^*}{\partial t} + \nabla\cdot(D^*\mathbf{u}) = 0\,, \quad \Rightarrow \quad \nabla\cdot\big(\rho_s(z)\,\theta_s(z)\mathbf{u}\big) = 0\,, \tag{3.19}$$

obtained via time independence of D^*, and the advection equations for specific entropy θ and density D,

$$\frac{\partial\theta}{\partial t} + \mathbf{u}\cdot\nabla\theta = 0\,, \qquad \frac{\partial D}{\partial t} + \nabla\cdot(D\mathbf{u}) = 0\,. \tag{3.20}$$

The PIA equations are the same as for the anelastic model except for the effects of the differently weighted incompressibility constraint in (3.19).

4 The dynamical anelastic-alpha equations are Euler–Poincaré

The Lagrangian for the anelastic-alpha model

Following Holm, Marsden and Ratiu [1998a,b], we introduce into the dynamical anelastic equations the effects of averaging over rapid fluctuations whose amplitude falls below the length scale denoted as α, by making the following modification of the Lagrangian in equation (3.1) for the anelastic model

$$\begin{aligned} L \;=\; \int d^3x\, \bigg[D\bigg(\frac{1}{2}|\mathbf{u}|^2 + \frac{\alpha^2}{2}|\nabla\mathbf{u}|^2 + \mathbf{u}\cdot\mathbf{R}(\mathbf{x}) - gz - C_p\pi_s(z)b\bigg) \\ + p'\bigg(1 - \frac{D}{\rho_s(z)}\bigg)\bigg]\,, \end{aligned} \tag{4.1}$$

where $|\nabla\mathbf{u}|^2 \equiv \mathbf{u}_{,k}\cdot\mathbf{u}_{,k}$ and we denote the other variables as before, in equation (3.1). This modified Lagrangian for the anelastic-alpha model produces the following variations at fixed $\mathbf{x}$ and t

$$\begin{aligned} &\frac{1}{D}\frac{\delta L}{\delta\mathbf{u}} = \mathbf{u} - \frac{\alpha^2}{D}\big(D\mathbf{u}_{,k}\big)_{,k} + \mathbf{R}(\mathbf{x})\,, \quad \frac{\delta L}{\delta b} = -C_p\pi_s(z)\,, \quad \frac{\delta L}{\delta p'} = 1 - \frac{D}{\rho_s(z)}\,, \\ &\frac{\delta L}{\delta D} = \frac{1}{2}|\mathbf{u}|^2 + \frac{\alpha^2}{2}|\nabla\mathbf{u}|^2 + \mathbf{u}\cdot\mathbf{R}(\mathbf{x}) - gz - C_p\pi_s(z)b - \frac{p'}{\rho_s(z)}\,. \end{aligned} \tag{4.2}$$

Hence, from the Euclidean component formula (2.1) for Hamilton principles of this type, the Euler–Poincaré equation for the Lagrangian (4.1) is given by

$$\frac{d\mathbf{v}}{dt} + v_j\nabla u^j - \mathbf{u}\times\mathrm{curl}\mathbf{R} + \nabla\bigg(\frac{p'}{\rho_s} - \frac{1}{2}|\mathbf{u}|^2 - \frac{\alpha^2}{2}|\nabla\mathbf{u}|^2\bigg) - \frac{g\theta'}{\theta_s}\hat{\mathbf{z}} = 0\,. \tag{4.3}$$

This is the motion equation for the anelastic-alpha fluid in three dimensions. One should compare this motion equation with equation (3.8) for the standard anelastic fluid and note that the advective time derivative is still defined as $d/dt = \partial/\partial t + \mathbf{u}\cdot\nabla$, in terms of the fluid parcel transport velocity, $\mathbf{u}$. We have used the following additional notation in equation (4.3),

$$\mathbf{v} \equiv \mathbf{u} - \alpha^2\tilde{\Delta}\mathbf{u}\,, \quad \text{and} \quad \tilde{\Delta}\mathbf{u} \equiv \frac{1}{D}\big(D\mathbf{u}_{,k}\big)_{,k}\,, \tag{4.4}$$

where the weighted Laplacian operator $\tilde{\Delta}$ is given by

$$\tilde{\Delta} = \frac{1}{\rho_s(z)}\left(\frac{\partial}{\partial x^k}\rho_s(z)\frac{\partial}{\partial x^k}\right) \quad \text{for} \quad D = \rho_s(z)\,, \tag{4.5}$$

and subscript-comma index notation denotes partial spatial derivatives. The key observation about the anelastic-alpha model is that its transport velocity $\mathbf{u}$ is smoother relative to its momentum or circulation velocity $\mathbf{v}$, by inversion of the Helmholtz operator, $(1-\tilde{\Delta})$, which depends upon the stratification profile of the base state. The curl form of equation (4.3) is

$$\frac{\partial \mathbf{v}}{\partial t} - \mathbf{u}\times \mathrm{curl}\,(\mathbf{R}+\mathbf{v}) + \nabla\left(\frac{p'}{\rho_s} - \frac{1}{2}|\mathbf{u}|^2 - \frac{\alpha^2}{2}|\nabla\mathbf{u}|^2 + \mathbf{u}\cdot\mathbf{v}\right) - \frac{g\theta'}{\theta_s}\hat{\mathbf{z}} = 0\,. \tag{4.6}$$

As we shall see, the analysis of the anelastic-alpha model follows the same procedure as for the anaelastic model, with appropriate changes from $\mathbf{u}$ to $\mathbf{v}$ to account for the effects of the averaging over the subgrid (sub-α-length) scale rapid fluctuations.

The Kelvin–Noether theorem for the anelastic-alpha model

From equation (2.5), the Kelvin–Noether circulation theorem corresponding to the anelastic motion equation (3.8) for an ideal anelastic-alpha model in three dimensions is,

$$\frac{d}{dt}\oint_{\gamma_t(\mathbf{u})}(\mathbf{v}+\mathbf{R})\cdot d\mathbf{x} = -\oint_{\gamma_t(\mathbf{u})} C_p\,\theta\,\nabla\pi_s(z)\cdot d\mathbf{x}\,, \tag{4.7}$$

where the curve $\gamma_t(\mathbf{u})$ follows the fluid parcels in moving with anelastic-alpha fluid velocity $\mathbf{u}$. Thus, by Stokes' theorem, the anelastic-alpha equations generate circulation of $(\mathbf{v}+\mathbf{R})$ around $\gamma_t(\mathbf{u})$ whenever the gradient of specific entropy θ has a horizontal component. Using advection of θ and the anelastic divergence condition, one finds conservation of the anelastic-alpha potential vorticity $q_{\mathrm{A}-\alpha}$ on fluid parcels, which are transported with the anelastic velocity $\mathbf{u}$, cf. equation (2.6),

$$\frac{\partial q_{\mathrm{A}-\alpha}}{\partial t} + \mathbf{u}\cdot\nabla q_{\mathrm{A}-\alpha} = 0\,, \quad \text{where} \quad q_{\mathrm{A}-\alpha} = \frac{1}{\rho_s(z)}\nabla\theta\cdot\mathrm{curl}\,(\mathbf{v}+\mathbf{R})\,. \tag{4.8}$$

Consequently, the following domain integrated quantities are conserved, for any function Φ, cf. equation (2.7),

$$C_\Phi = \int d^3x\ \rho_s(z)\,\Phi(\theta, q_{\mathrm{A}-\alpha})\,, \quad \forall\,\Phi\,. \tag{4.9}$$

Energy conservation for the anelastic-alpha model

The conserved anelastic-alpha energy is calculated as before using the Legendre transform of the Lagrangian (4.1) and found to be

$$E_{\mathrm{A}-\alpha} = \int d^3x\ \rho_s(z)\left(\frac{1}{2}|\mathbf{u}|^2 + \frac{\alpha^2}{2}|\nabla\mathbf{u}|^2 + gz + C_p\pi_s(z)\theta\right). \tag{4.10}$$

Thus, the kinetic energy is augmented in the anelastic-alpha model by a term proportional to the squared amplitude of the velocity shear. Hence, the anelastic-alpha model costs energy to produce velocity shear and its solutions will tend to have smoother velocity profiles than those for the standard anelastic model.

The curl of equation (4.6) and the anelastic divergence condition $\nabla \cdot \rho_s \mathbf{u} = 0$ yield an equation for anelastic-alpha vortex dynamics,

$$\frac{\partial \mathbf{q}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{q} = \mathbf{q} \cdot \nabla \mathbf{u} + \frac{1}{\rho_s} \nabla \left(\frac{\theta'}{\theta_s} \right) \times g\hat{\mathbf{z}} \,, \quad \text{where} \quad \mathbf{q} \equiv \frac{1}{\rho_s(z)} \text{curl}(\mathbf{v} + \mathbf{R}) \,. \tag{4.11}$$

The control on the L^2 norm $\|\nabla \mathbf{u}\|_2$ afforded by the conserved energy in equation (4.10) should tend to moderate the vortex stretching term $\mathbf{q} \cdot \nabla \mathbf{u}$ in the vortex dynamics equation (4.11) and, thus, produce less violent, smoother and more coherent turbulent vortex dynamics than occurs for the standard anelastic equations. Numerical investigation of the anelastic-alpha system introduced here will be conducted and reported elsewhere.

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